Solid Mechanics - 202041

Mr. K. B. Bansode

Unit V Principal Stresses, Theories of Failure

CO5. APPLY the concept of principal stresses and theories of failure to determine stresses on a 2-D element.

Principal Stresses: Introduction to principal stresses with application, Transformation of Plane Stress, Principal Stresses and planes (Analytical method and Mohr's Circle), Stresses due to combined Normal and Shear stresses

Theories of Elastic failure: Introduction to theories of failure with application, Maximum principal stress theory, Maximum shear stress theory, Maximum distortion energy theory, Maximum principal strain theory, Maximum strain energy theory

The planes, which have no shear stress, are known as principal planes. Hence principal planes are the planes of zero shear stress. These planes carry only normal stresses.

The normal stresses, acting on a principal plane, are known as principal stresses.

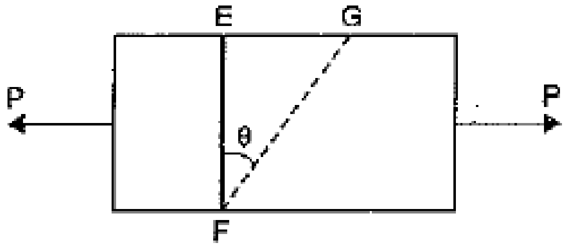


Fig. 3.1 (*a*)

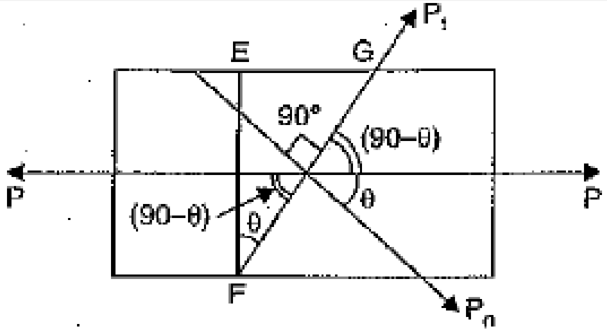


Fig. 3.2 (*b*)

Then area of section, $EF = EF \times 1 = A.$

The stress on the section *EF* is given by

$$\sigma = \frac{\text{Force}}{\text{Area of } EF} = \frac{P}{A} \qquad \ldots(i)$$

The stress on the section *EF* is entirely normal stress. There is no shear stress (or tangential stress) on the section *EF*.

Now consider a section *FG* at an angle θ with the normal cross-section *EF* as shown in Fig. 3.1 (*a*).

Area of section $FG = FG \times 1$ (member is having unit thickness)

$$= \frac{EF}{\cos\theta} \times 1 \qquad \left(\because \text{In } \Delta\, EFG, \frac{EF}{FG} = \cos\theta \;\therefore FG = \frac{EF}{\cos\theta}\right)$$

$$= \frac{A}{\cos\theta} \qquad (\because \; EF \times 1 = A)$$

∴ Stress on the section, FG

$$= \frac{\text{Force}}{\text{Area of section } FG} = \frac{P}{\left(\frac{A}{\cos\theta}\right)} = \frac{P}{A}\cos\theta$$

$$= \sigma\cos\theta \qquad \left(\because \frac{P}{A} = \sigma\right) \qquad \ldots(3.1)$$

This stress, on the section FG, is parallel to the axis of the member (*i.e.*, this stress is along x-axis). This stress may be resolved in two components. One component will be normal to the section FG whereas the second component will be along the section FG (*i.e.*, tangential to the section FG). The normal stress and tangential stress (*i.e.*, shear stress) on the section FG are obtained as given below [Refer to Fig. 3.1 (*b*)].

Let P_n = The component of the force P, normal to section FG

$= P\cos\theta$

P_t = The component of force P, along the surface of the section FG (or tangential to the surface FG)

$= P\sin\theta$

σ_n = Normal stress across the section FG

σ_t = Tangential stress (*i.e.*, shear stress) across the section FG.

∴ Normal stress and tangential stress across the section FG are obtained as,

Normal stress,

$$\sigma_n = \frac{\text{Force normal to section } FG}{\text{Area of section } FG}$$

$$= \frac{P_n}{\left(\frac{A}{\cos\theta}\right)} = \frac{P\cos\theta}{\left(\frac{A}{\cos\theta}\right)} \qquad (\because \quad P_n = P\cos\theta)$$

$$= \frac{P}{A}\cos\theta \,.\, \cos\theta = \frac{P}{A}\cos^2\theta$$

$$= \sigma\cos^2\theta \qquad \left(\because \frac{P}{A} = \sigma\right) \qquad \ldots(3.2)$$

Tangential stress (*i.e.*, shear stress),

$$\sigma_t = \frac{\text{Tangential force across section } FG}{\text{Area of section } FG}$$

$$= \frac{P_t}{\left(\frac{A}{\cos\theta}\right)} = \frac{P\sin\theta}{\left(\frac{A}{\cos\theta}\right)} \qquad (\because \quad P_t = P\sin\theta)$$

$$= \frac{P}{A}\sin\theta \,.\, \cos\theta = \sigma\sin\theta .\, \cos\theta$$

$$= \frac{\sigma}{2} \times 2 \sin\theta \cos\theta \qquad \text{[Multiplying and dividing by 2]}$$

$$= \frac{\sigma}{2} \sin 2\theta \qquad (\because \ 2 \sin\theta \cos\theta = \sin 2\theta) \qquad ...(3.3)$$

From equation (3.2), it is seen that the normal stress (σ_n) on the section *FB* will be maximum, when $\cos^2\theta$ or $\cos\theta$ is maximum. And $\cos\theta$ will be maximum when $\theta = 0°$ as $\cos 0° = 1$. But when $\theta = 0°$, the section *FG* will coincide with section *EF*. But the section *EF* is normal to the line of action of the loading. This means the plane normal to the axis of loading will carry the maximum normal stress.

$$\therefore \text{ Maximum normal stress, } = \sigma \cos^2\theta = \sigma \cos^2 0° = \sigma \qquad ...(3.4)$$

From equation (3.3), it is observed that the tangential stress (*i.e.*, shear stress) across the section *FG* will be maximum when $\sin 2\theta$ is maximum. And $\sin 2\theta$ will be maximum when $\sin 2\theta = 1$ or $2\theta = 90°$ or $270°$

or $\theta = 45°$ or $135°$.

This means the shear stress will be maximum on two planes inclined at 45° and 135° to the normal section *EF* as shown in Figs. 3.1 (*c*) and 3.1 (*d*).

$$\therefore \text{ Max. value of shear stress} = \frac{\sigma}{2} \sin 2\theta = \frac{\sigma}{2} \sin 90° = \frac{\sigma}{2}. \qquad ...(3.5)$$

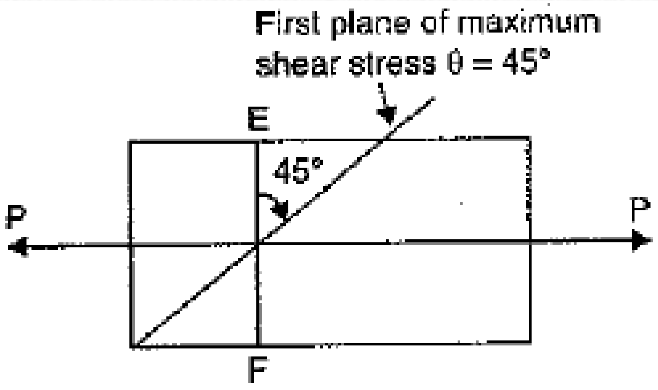


Fig. 3.1 (*c*)

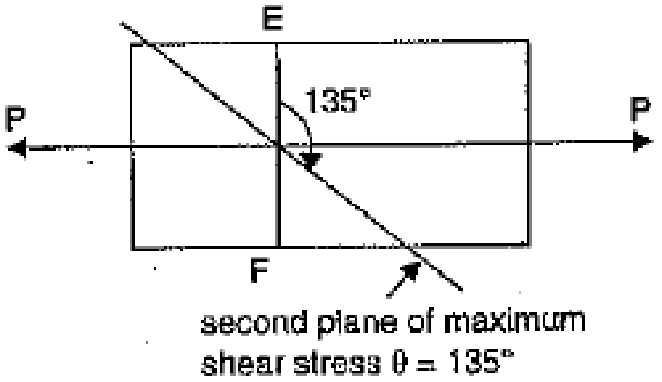


Fig. 3.1 (*d*)

From equations (3.4) and (3.5) it is seen that maximum normal stress is equal to σ whereas the maximum shear stress is equal to $\sigma/2$ or equal to half the value of greatest normal stress.

Problem 3.1. *A rectangular bar of cross-sectional area 10000 mm² is subjected to an axial load of 20 kN. Determine the normal and shear stresses on a section which is inclined at an angle of 30° with normal cross-section of the bar.*

Sol. Given :

Cross-sectional area of the rectangular bar,

$$A = 10000 \text{ mm}^2$$

Axial load, $P = 20 \text{ kN} = 20{,}000 \text{ N}$

Angle of oblique plane with the normal cross-section of the bar,

$$\theta = 30°$$

Now direct stress, $\sigma = \dfrac{P}{A} = \dfrac{20000}{10000} = 2 \text{ N/mm}^2$

Let σ_n = Normal stress on the oblique plane

σ_t = Shear stress on the oblique plane.

Using equation (3.2) for normal stress, we get

$$\sigma_n = \sigma \cos^2 \theta$$

$$= 2 \times \cos^2 30° \qquad (\because \sigma = 2 \text{ N/mm}^2)$$

$$= 2 \times 0.866^2 \qquad (\because \cos 30° = 0.866)$$

$$= \mathbf{1.5 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Using equation (3.3) for shear stress, we get

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta = \frac{2}{2} \times \sin (2 \times 30°)$$

$$= 1 \times \sin 60° = \mathbf{0.866 \text{ N/mm}^2}. \quad \textbf{Ans.}$$

Problem 3.2. *Find the diameter of a circular bar which is subjected to an axial pull of 160 kN, if the maximum allowable shear stress on any section is 65 N/mm².*

Sol. Given :

Axial pull, $P = 160 \text{ kN} = 160000 \text{ N}$

Maximum shear stress $= 65 \text{ N/mm}^2$

Let D = Diameter of the bar

$\therefore$ Area of the bar $= \frac{\pi}{4} D^2$

$\therefore$ Direct stress, $\sigma = \frac{P}{A} = \frac{160000}{\frac{\pi}{4} D^2} = \frac{640000}{\pi D^2} \text{ N/mm}^2$

Maximum shear stress is given by equation (3.5).

$\therefore$ Maximum shear stress $= \frac{\sigma}{2} = \frac{640000}{2 \times \pi D^2}$.

But maximum shear stress is given as = 65 N/mm².

Hence equating the two values of maximum shear, we get

$$\therefore \quad 65 = \frac{640000}{2 \times \pi D^2}$$

$$\therefore \quad D^2 = \frac{640000}{2 \times \pi \times 65} = 1567$$

$$\therefore \quad D = 39.58 \text{ mm.} \quad \textbf{Ans.}$$

Problem 3.3. *A rectangular bar of cross-sectional area of 11000 mm² is subjected to a tensile load P as shown in Fig. 3.3. The permissible normal and shear stresses on the oblique plane BC are given as 7 N/mm² and 3.5 N/mm² respectively. Determine the safe value of P.*

Sol. Given :

Area of cross-section, $A = 11000 \text{ mm}^2$

Normal stress, $\sigma_n = 7 \text{ N/mm}^3$

Shear stress, $\sigma_t = 3.5 \text{ N/mm}^2$

Angle of oblique plane with the axis of bar = 60°.

∴ Angle of oblique plane *BC* with the normal cross-section of the bar,

$$\theta = 90° - 60° = 30°$$

Let P = Safe value of axial pull

σ = Safe stress in the member.

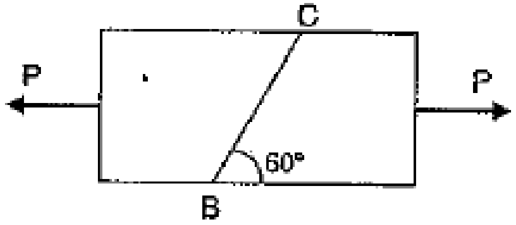


Fig. 3.3

Using equation (3.2),

$$\sigma_n = \sigma \cos^2 \theta \quad \text{or} \quad 7 = \sigma \cos^2 30°$$

$$= \sigma (0.866)^2. \qquad (\because \cos 30° = 0.866)$$

∴
$$\sigma = \frac{7}{0.866 \times 0.866} = 9.334 \text{ N/mm}^2$$

Using equation (3.3),

$$\sigma_t = \frac{\sigma}{2} \sin 2\theta$$

or
$$3.5 = \frac{\sigma}{2} \sin 2 \times 30^\circ = \frac{\sigma}{2} \sin 60^\circ = \frac{\sigma}{2} \times 0.866$$

$$\therefore \quad \sigma = \frac{3.5 \times 2}{0.866} = 8.083 \text{ N/mm}^2.$$

The safe stress is the least of the two, *i.e.*, 8.083 N/mm^2.

$\therefore$ Safe value of axial pull,

$$P = \text{Safe stress} \times \text{Area of cross-section}$$
$$= 8.083 \times 11000 = 88913 \text{ N} = \mathbf{88.913 \text{ kN}}. \quad \textbf{Ans.}$$

3.4.2. A Member Subjected to like Direct Stresses in two Mutually Perpendicular Directions. Fig. 3.4 (*a*) shows a rectangular bar *ABCD* of uniform cross-sectional area *A* and of unit thickness. The bar is subjected to two direct tensile stresses (or two-principal tensile stresses) as shown in Fig. 3.4 (*a*).

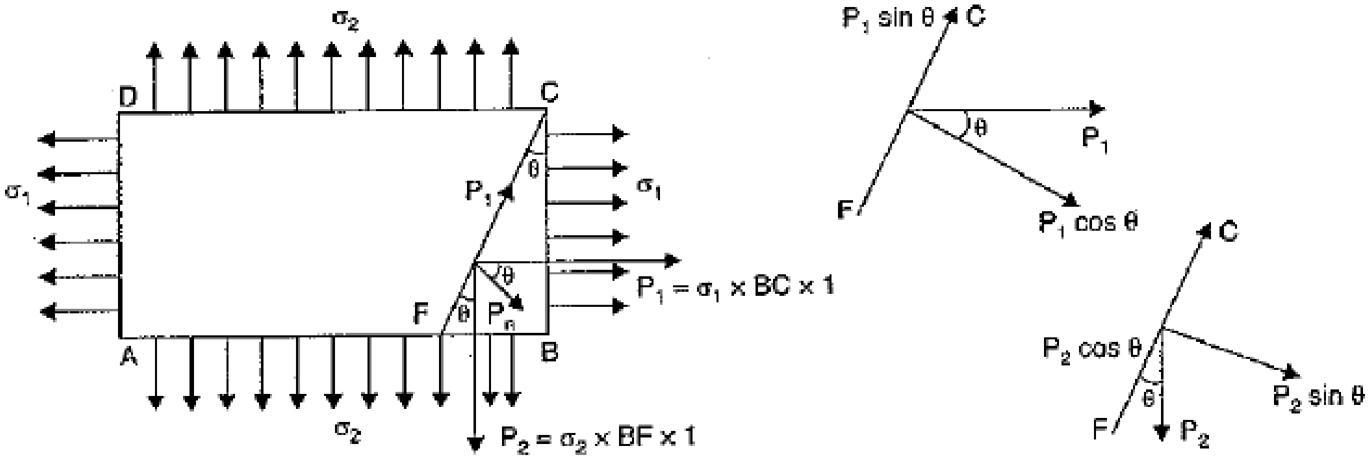


Fig. 3.4 (*a*)

Let *FC* be the oblique section on which stresses are to be calculated. This can be done by converting the stresses σ_1 (acting on face *BC*) and σ_2 (acting on face *AB*) into equivalent forces. Then these forces will be resolved along the inclined plane *FC* and perpendicular to *FC*. Consider the forces acting on wedge *FBC*.

Let θ = Angle made by oblique section *FC* with normal cross-section *BC*

σ_1 = Major tensile stress on face *AD* and *BC*

σ_2 = Minor tensile stress on face *AB* and *CD*

P_1 = Tensile force on face BC

P_2 = Tensile force on face FB.

The tensile force on face BC,

$$P_1 = \sigma_1 \times \text{Area of face } BC = \sigma_1 \times BC \times 1 \qquad (\because \text{ Area} = BC \times 1)$$

The tensile force on face FB,

$$P_2 = \text{Stress on } FB \times \text{Area of } FB = \sigma_2 \times FB \times 1.$$

The tensile forces P_1 and P_2 are also acting on the oblique section FC. The force P_1 is acting in the axial direction, whereas the force P_2 is acting downwards as shown in Fig. 3.4 (*a*). Two forces P_1 and P_2 each can be resolved into two components *i.e.*, one normal to the plane FC and other along the plane FC. The components of P_1 are $P_1 \cos\theta$ normal to the plane FC and $P_1 \sin\theta$ along the plane in the upward direction. The components of P_2 are $P_2 \sin\theta$ normal to the plane FC and $P_2 \cos\theta$ along the plane in the downward direction.

Let P_n = Total force normal to section FC

= Component of force P_1 normal to section FC + Component of force P_2 normal to section FC

$$= P_1 \cos\theta + P_2 \sin\theta$$

$$= \sigma_1 \times BC \times \cos\theta + \sigma_2 \times BF \times \sin\theta \qquad (\because P_1 = \sigma_1 \times BC,\ P_2 = \sigma_2 \times BF)$$

P_t = Total force along the section FC

$= $ Component of force P_1 along the section FC

$+$ Component of force P_2 along the section FC

$= P_1 \sin\theta + (-P_2 \cos\theta)$ (–ve sign is taken due to opposite direction)

$= P_1 \sin\theta - P_2 \cos\theta$

$= \sigma_1 \times BC \times \sin\theta - \sigma_2 \times BF \times \cos\theta$

(Substituting the values P_1 and P_2)

$\sigma_n = $ Normal stress across the section FC

$$= \frac{\text{Total force normal to the section } FC}{\text{Area of section } FC}$$

$$= \frac{P_n}{FC \times 1} = \frac{\sigma_1 \times BC \times \cos\theta + \sigma_2 \times BF \times \sin\theta}{FC}$$

$$= \sigma_1 \times \frac{BC}{FC} \times \cos\theta + \sigma_2 \times \frac{BF}{FC} \times \sin\theta$$

$$= \sigma_1 \times \cos\theta \times \cos\theta + \sigma_2 \times \sin\theta \times \sin\theta$$

$$\left(\because \text{ In triangle } FBC,\ \frac{BC}{FC} = \cos\theta,\ \frac{BF}{FC} = \sin\theta\right)$$

$$= \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta$$

$$= \sigma_1 \left(\frac{1 + \cos 2\theta}{2}\right)^{*} + \sigma_2 \left(\frac{1 - \cos 2\theta}{2}\right)^{**}$$

$[\because \cos^2\theta = (1 + \cos 2\theta)/2$ and $\sin^2\theta = (1 - \cos 2\theta)/2]$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta \qquad \text{...(3.6)}$$

σ_t = Tangential stress (or shear stress) along section FC

$$= \frac{\text{Total force along the section } FC}{\text{Area of section } FC} \qquad \left(\because \text{ Stress} = \frac{\text{Force}}{\text{Area}}\right)$$

$$^{*} \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$
$$\therefore \quad \cos^2\theta = \frac{(1 + \cos 2\theta)}{2}$$

$$^{**} \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$
$$\therefore \quad \sin^2\theta = \frac{(1 - \cos 2\theta)}{2}$$

$$= \frac{P_t}{FC \times 1} = \frac{\sigma_1 \times BC \times \sin\theta - \sigma_2 \times BF \times \cos\theta}{FC}$$

$$= \sigma_1 \times \frac{BC}{FC} \times \sin\theta - \sigma_2 \times \frac{BF}{FC} \times \cos\theta$$

$$= \sigma_1 \times \cos\theta \times \sin\theta - \sigma_2 \times \sin\theta \times \cos\theta$$

$$\left(\because \text{ In triangle } FBC,\ \frac{BC}{FC} = \cos\theta,\ \frac{BF}{FC} = \sin\theta\right)$$

$$= (\sigma_1 - \sigma_2)\cos\theta\sin\theta$$

$$= \frac{(\sigma_1 - \sigma_2)}{2} \times 2\cos\theta\sin\theta \qquad \text{(Multiplying and dividing by 2)}$$

$$= \frac{(\sigma_1 - \sigma_2)}{2}\sin 2\theta \qquad \text{...(3.7)}$$

The resultant stress on the section FC will be given as

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2} \qquad \text{...(3.8)}$$

Obliquity [Refer to Fig. 3.4 (*b*)]. The angle made by the resultant stress with the normal of the oblique plane, is known as obliquity. It is denoted by ϕ. Mathematically,

$$\tan\phi = \frac{\sigma_t}{\sigma_n} \qquad \text{...[3.8 (A)]}$$

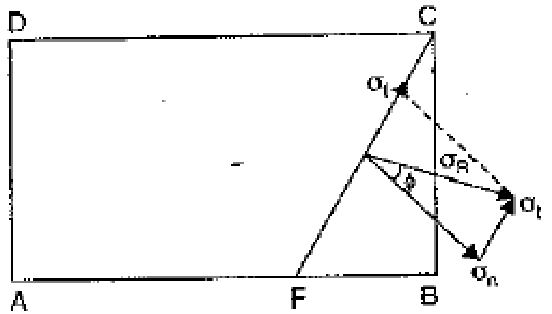


Fig. 3.4 (*b*)

Maximum shear stress. The shear stress is given by equation (3.7). The shear stress (σ_t) will be maximum when

$$\sin 2\theta = 1 \quad \text{or} \quad 2\theta = 90° \quad \text{or} \quad 270° \quad (\because \quad \sin 90° = 1 \text{ and also } \sin 270° = 1)$$

or $$\theta = 45° \text{ or } 135°$$

And maximum shear stress, $$(\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2} \qquad ...(3.9)$$

The planes of maximum shear stress are obtained by making an angle of 45° and 135° with the plane *BC* (at any point on the plane *BC*) in such a way that the planes of maximum shear stress lie within the material as shown in Fig. 3.4 (*c*).

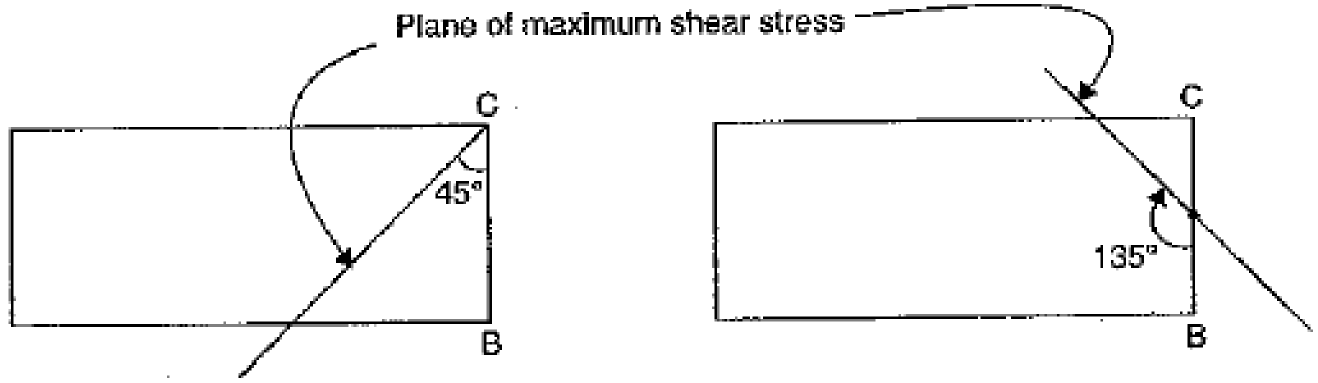


Fig. 3.4 (*c*)

Hence the planes, which are at an angle of 45° or 135° with the normal cross-section *BC* [see Fig. 3.4 (*c*)], carry the maximum shear stresses.

Principal planes. Principal planes are the planes on which shear stress is zero. To locate the position of principal planes, the shear stress given by equation (3.7) should be equated to zero.

$\therefore$ For principal planes,

$$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = 0$$

or $\quad \sin 2\theta = 0 \qquad [\because \ (\sigma_1 - \sigma_2) \text{ cannot be equal to zero}]$

or $\quad 2\theta = 0 \ \text{ or } \ 180°$

$\therefore \quad \theta = 0 \ \text{ or } \ 90°$

when $\theta = 0$,

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 0°$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times 1 \qquad (\because \ \cos 0° = 1)$$

$$= \sigma_1$$

when $\theta = 90°$,

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2 \times 90°$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 180°$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times (-1) \qquad (\because \ \cos 180° = -1)$$

$$= \sigma_2.$$

3.4.3. A Member Subjected to a Simple Shear Stress. Fig. 3.8 shows a rectangular bar *ABCD* of uniform cross-sectional area *A* and of unit thickness. The bar is subjected to a simple shear stress (q) across the faces *BC* and *AD*. Let *FC* be the oblique section on which normal and tangential stresses are to be calculated.

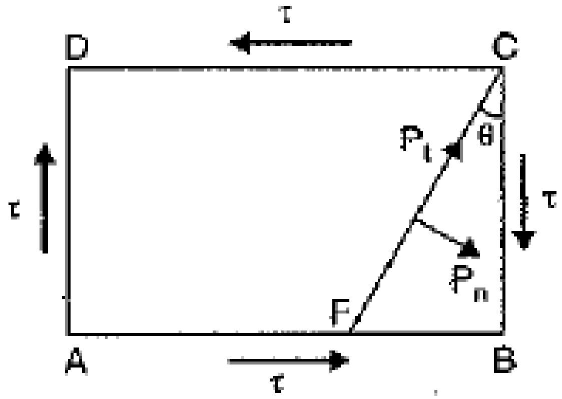


Fig. 3.8

Let θ = Angle made by oblique section *FC* with normal cross-section *BC*,

τ = Shear stress across faces *BC* and *AD*.

It has already been proved (Refer Art. 2.9) that a shear stress is always accompanied by an equal shear stress at right angles to it. Hence the faces *AB* and *CD* will also be subjected to a shear stress q as shown in Fig. 3.8. Now these stresses will be converted into equivalent forces. Then these forces will be resolved along the inclined surface and normal to inclined surface. Consider the forces acting on the wedge *FBC* of Fig. 3.9.

Let Q_1 = Shear force on face BC

= Shear stress × Area of face BC

$= \tau \times BC \times 1$

($\because$ Area of face $BC = BC \times 1$)

$= \tau \times BC$

Q_2 = Shear force on face FB

$= \tau \times$ Area of FB

$= \tau \times FB \times 1 = \tau \cdot FB$

P_n = Total normal force on section FC

P_t = Total tangential force on section FC.

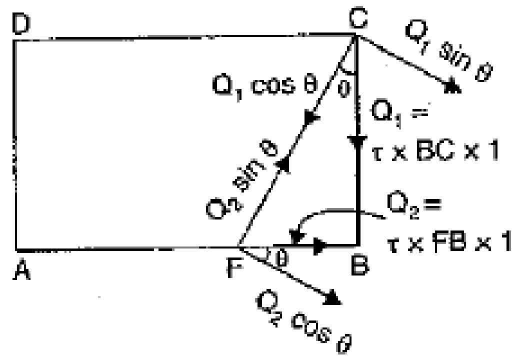


Fig. 3.9

The force Q_1 is acting along face CB as shown in Fig. 3.9. This force is resolved into two components *i.e.*, $Q_1 \cos\theta$ and $Q_1 \sin\theta$ along the plane CF and normal to the plane CF respectively.

The force Q_2 is acting along the face FB. This force is also resolved into two component *i.e.*, $Q_2 \sin\theta$ and $Q_2 \cos\theta$ along the plane FC and normal to the plane FC respectively.

$\therefore$ Total normal force on section FC,

$$P_n = Q_1 \sin\theta + Q_2 \cos\theta$$

$$= \tau \times BC \times \sin\theta + \tau \times FB \times \cos\theta. \qquad (\because \; Q_1 = \tau \times BC \text{ and } Q_2 = \tau \times FB)$$

And total tangential force on section FC.

$$P_t = Q_2 \sin\theta - Q_1 \cos\theta. \qquad \text{(–ve sign is taken due to opposite direction)}$$

$$= \tau \times FB \times \sin\theta - \tau \times BC \times \cos\theta \qquad (\because \; Q_2 = \tau \cdot FB \text{ and } Q_1 = \tau \cdot BC)$$

Let σ_n = Normal stress on section FC

σ_t = Tangential stress on section FC

Then $\sigma_n = \dfrac{\text{Total normal force on section } FC}{\text{Area of section } FC}$

$$= \frac{P_n}{FC \times 1}$$

$$= \frac{\tau . BC . \sin\theta + \tau . FB . \cos\theta}{FC \times 1} \qquad (\because \text{ Area} = FC \times 1)$$

$$= \tau . \frac{BC}{FC} . \sin\theta + \tau . \frac{FB}{FC} . \cos\theta$$

$$= \tau . \cos\theta . \sin\theta + \tau . \sin\theta . \cos\theta$$

$$\left(\because \text{ In triangle } FBC, \frac{BC}{FC} = \cos\theta, \frac{FB}{FC} = \sin\theta\right)$$

$$= 2\tau \cos\theta . \sin\theta$$

$$= \tau \sin 2\theta \qquad (\because \ 2 \sin\theta \cos\theta = \sin 2\theta) \qquad ...(3.10)$$

and $\sigma_t = \dfrac{\text{Total tangential force on section } FC}{\text{Area of section } FC}$

$$= \frac{P_t}{FC \times 1}$$

$$= \frac{\tau \times FB \times \sin\theta - \tau \times BC \times \cos\theta}{FC \times 1}$$

$$= \tau \times \frac{FB}{FC} \times \sin\theta - \tau \times \frac{BC}{FC} \times \cos\theta$$

$$= \tau \times \sin\theta \times \sin\theta - \tau \times \cos\theta \times \cos\theta$$

$$= \tau \sin^2 \theta - \tau \cos^2 \theta = -\tau [\cos^2 \theta - \sin^2 \theta]$$

$$= -\tau \cos 2\theta \qquad (\because \quad \cos^2 \theta - \sin^2 \theta = \cos 2\theta) \quad \ldots(3.11)$$

–ve sign shows that σ_t will be acting downwards on the plane *CF*.

A Member Subjected to Direct Stresses in two Mutually Perpendicular Directions Accompanied by a Simple Shear Stress. Fig. 3.10 (*a*) shows a rectangular bar *ABCD* of uniform cross-sectional area *A* and of unit thickness. This bar is subjected to :

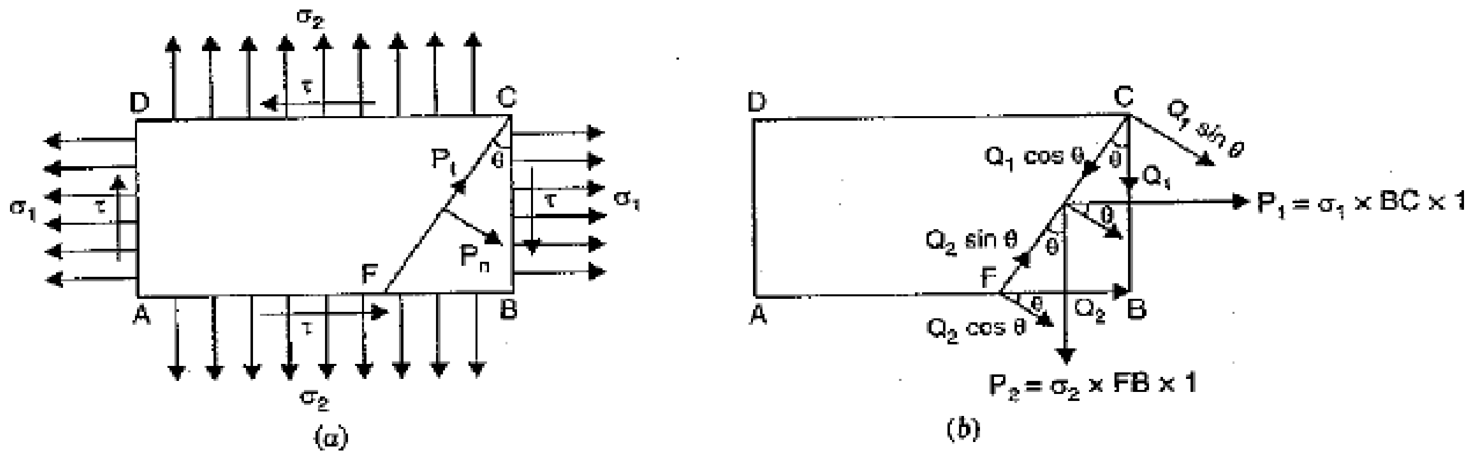


Fig. 3.10

(*i*) tensile stress σ_1 on the face *BC* and *AD*

(*ii*) tensile stress σ_2 on the face *AB* and *CD*

(*iii*) a simple shear stress τ on face *BC* and *AD*.

But with reference to Art. 2.9, a simple shear stress is always accompanied by an equal shear stress at right angles to it. Hence the faces *AB* and *CD* will also be subjected to a shear stress τ as shown in Fig. 3.10 (*a*).

We want to calculate normal and tangential stresses on oblique section *FC*, which is inclined at an angle θ with the normal cross-section *BC*. The given stresses are converted into equivalent forces.

The forces acting on the wedge FBC are :

P_1 = Tensile force on face BC due to tensile stress σ_1

$= \sigma_1 \times$ Area of BC

$= \sigma_1 \times BC \times 1$ $\quad$ ($\because$ Area $= BC \times 1$)

$= \sigma_1 \times BC$

P_2 = Tensile force on face FB due to tensile stress σ_2

$= \sigma_2 \times$ Area of $FB = \sigma_2 \times FB \times 1$

$= \sigma_2 \times FB$

Q_1 = Shear force on face BC due to shear stress τ

$= \tau \times$ Area of BC

$= \tau \times BC \times 1 = \tau \times BC$

Q_2 = Shear force on face FB due to shear stress τ

$= \tau \times$ Area of FB

$= \tau \times FB \times 1 = \tau \times FB.$

Resolving the above four forces (*i.e.*, P_1, P_2, Q_1 and Q_2) normal to the oblique section FC, we get

Total normal force,

$$P_n = P_1 \cos\theta + P_2 \sin\theta + Q_1 \sin\theta + Q_2 \cos\theta$$

Substituting the values of P_1, P_2, Q_1 and Q_2, we get

$$P_n = \sigma_1 . BC . \cos\theta + \sigma_2 . FB . \sin\theta + \tau . BC . \sin\theta + \tau . FB . \cos\theta$$

Similarly, the total tangential force (P_t) is obtained by resolving P_1, P_2, Q_1 and Q_2 along the oblique section FC.

∴ Total tangential force,

$$P_t = P_1 \sin\theta - P_2 \cos\theta - Q_1 \cos\theta + Q_2 \sin\theta$$

$$= \sigma_1 . BC . \sin\theta - \sigma_2 . FB . \cos\theta - \tau . BC . \cos\theta + \tau . FB . \sin\theta$$

(substitute the values of P_1, P_2, Q_1 and Q_2)

Now, Let σ_n = Normal stress across the section FC, and

σ_t = Tangential stress across the section FC.

Then normal stress across the section FC,

$$\sigma_n = \frac{\text{Total normal force across section } FC}{\text{Area of section } FC} = \frac{P_n}{FC \times 1}$$

$$= \frac{\sigma_1 . BC . \cos\theta + \sigma_2 . FB . \sin\theta + \tau . BC . \sin\theta + \tau . FB . \cos\theta}{FC \times 1}$$

$$= \sigma_1 . \frac{BC}{FC} . \cos\theta + \sigma_2 . \frac{FB}{FC} . \sin\theta + \tau . \frac{BC}{FC} . \sin\theta + \tau . \frac{FB}{FC} . \cos\theta$$

$$= \sigma_1 . \cos\theta . \cos\theta + \sigma_2 \sin\theta . \sin\theta + \tau . \cos\theta . \sin\theta + \tau \sin\theta . \cos\theta$$

$$\left(\because \text{ In triangle } FBC, \frac{BC}{FC} = \cos\theta \text{ and } \frac{FB}{FC} = \sin\theta\right)$$

$$= \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta + 2\tau \cos\theta \sin\theta$$

$$= \sigma_1 \left(\frac{1+\cos 2\theta}{2}\right) + \sigma_2 \left(\frac{1-\cos 2\theta}{2}\right) + \tau \sin 2\theta$$

$$\left(\because \cos^2\theta = \frac{1+\cos 2\theta}{2}, \sin^2\theta = \frac{1-\cos 2\theta}{2} \text{ and } 2\cos\theta\sin\theta = \sin 2\theta\right)$$

$$= \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta + \tau \sin 2\theta \qquad \text{...(3.12)}$$

and tangential stress (*i.e.*, shear stress) across the section FC,

$$\sigma_t = \frac{\text{Total tangential force across section } FC}{\text{Area of section } FC} = \frac{P_t}{FC \times 1}$$

$$= \frac{\sigma_1 . BC . \sin\theta - \sigma_2 . FB . \cos\theta - \tau . BC . \cos\theta + \tau . FB . \sin\theta}{FC \times 1}$$

$$= \sigma_1 . \frac{BC}{FC} . \sin\theta - \sigma_2 . \frac{FB}{FC} . \cos\theta - \tau . \frac{BC}{FC} . \cos\theta + \tau . \frac{FB}{FC} . \sin\theta$$

$$= \sigma_1 . \cos\theta . \sin\theta - \sigma_2 . \sin\theta . \cos\theta - \tau . \cos\theta . \cos\theta + \tau . \sin\theta . \sin\theta$$

$$\left(\because \text{In triangle } FBC, \frac{BC}{FC} = \cos\theta \text{ and } \frac{FB}{FC} = \sin\theta\right)$$

$$= (\sigma_1 - \sigma_2) . \cos\theta \sin\theta - \tau \cos^2\theta + \tau \sin^2\theta$$

$$= \left(\frac{\sigma_1-\sigma_2}{2}\right) . 2\cos\theta\sin\theta - \tau(\cos^2\theta - \sin^2\theta)$$

Position of principal planes. The planes on which shear stress (*i.e.*, tangential stress) is zero, are known as principal planes. And the stresses acting on principal planes are known principal stresses.

The position of principal planes are obtained by equating the tangential stress [given by equation (3.13)] to zero.

$\therefore$ For principal planes, $\sigma_t = 0$

or $$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta = 0$$

or $$\frac{\sigma_1 - \sigma_2}{2} \sin 2\theta = \tau \cos 2\theta$$

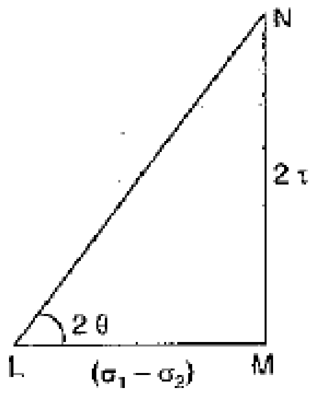
or $$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\tau}{\frac{(\sigma_1 - \sigma_2)}{2}} = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$

or $$\tan 2\theta = \frac{2\tau}{(\sigma_1 - \sigma_2)} \qquad \text{...(3.14)}$$

But the tangent of any angle in a right angled triangle

$$= \frac{\text{Height of right angled triangle}}{\text{Base of right angled triangle}}$$

$$\therefore \quad \frac{\text{Height of right angled triangle}}{\text{Base of right angled triangle}} = \frac{2\tau}{(\sigma_1 - \sigma_2)}$$



$\therefore$ Height of right angled triangle $= 2\tau$

Base of right angled triangle $= (\sigma_1 - \sigma_2)$.

Now diagonal of the right angled triangle

$$= \pm \sqrt{(\sigma_1 - \sigma_2)^2 + (2\tau)^2} = \pm \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

Fig. 3.11

$$= \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \quad \text{and} \quad -\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

1st Case. Diagonal $= \sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$

Then
$$\sin 2\theta = \frac{\text{Height}}{\text{Diagonal}} = \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

and
$$\cos 2\theta = \frac{\text{Base}}{\text{Diagonal}} = \frac{(\sigma_1 - \sigma_2)}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}.$$

The value of major principal stress is obtained by substituting the values of sin 2θ and cos 2θ in equation (3.12).

∴ Major principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times \frac{(\sigma_1 - \sigma_2)}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \tau \times \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{1}{2} \frac{(\sigma_1 - \sigma_2)^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \frac{2\tau^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{(\sigma_1 - \sigma_2)^2 + 4\tau^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2} \qquad \text{...(3.15)}$$

2nd Case. Diagonal $= -\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$

Then $$\sin 2\theta = \frac{2\tau}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

and $$\cos 2\theta = \frac{(\sigma_1 - \sigma_2)}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

Substituting these values in equation (3.12), we get minor principal stress.

$\therefore$ Minor principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \times \frac{\sigma_1 - \sigma_2}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} + \tau \times \frac{2\tau}{-\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \frac{(\sigma_1 - \sigma_2)^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}} - \frac{2\tau^2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \frac{(\sigma_1 - \sigma_2)^2 + 4\tau^2}{2\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{(\sigma_1 - \sigma_2)}{2}\right)^2 + \tau^2} \qquad \text{...(3.16)}$$

Equation (3.15) gives the maximum principal stress whereas equation (3.16) gives minimum principal stress. These two principal planes are at right angles.

The position of principal planes is obtained by finding two values of θ from equation (3.14).

Fig. 3.11(a) shows the principal planes in which θ_1 and θ_2 are the values from equation (3.14).

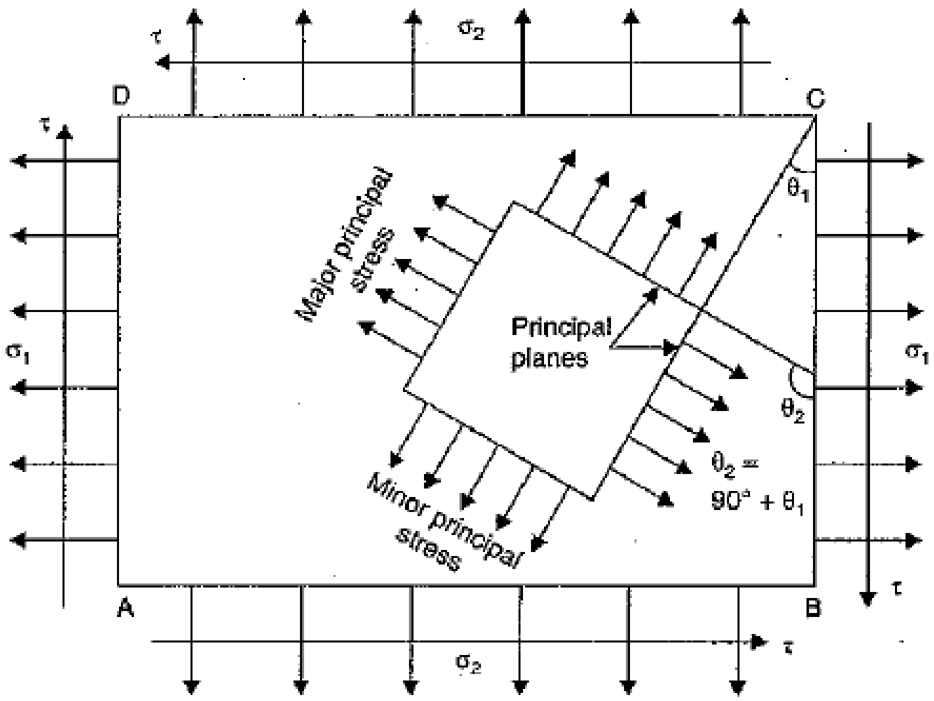


Fig. 3.11 (*a*)

Maximum shear stress. The shear stress is given by equation (3.13). The shear stress will be maximum or minimum when

$$\frac{d}{d\theta}(\sigma_t) = 0$$

or $$\frac{d}{d\theta}\left[\frac{\sigma_1 - \sigma_2}{2}\sin 2\theta - \tau\cos 2\theta\right] = 0$$

or $$\frac{\sigma_1 - \sigma_2}{2} (\cos 2\theta) \times 2 - \tau(-\sin 2\theta) \times 2 = 0$$

$$(\sigma_1 - \sigma_2) \,.\, \cos 2\theta + 2\tau \sin 2\theta = 0$$

or $$2\tau \sin 2\theta = -(\sigma_1 - \sigma_2) \cos 2\theta$$
$$= (\sigma_2 - \sigma_1) \cos 2\theta$$

or $$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\sigma_2 - \sigma_1}{2\tau}$$

or $$\tan 2\theta = \frac{\sigma_2 - \sigma_1}{2\tau} \quad \text{...(3.17)}$$

Equation (3.17) gives condition for maximum or minimum shear stress.

If $\tan 2\theta = \frac{\sigma_2 - \sigma_1}{2\tau}$

Then $$\sin 2\theta = \pm \frac{\sigma_2 - \sigma_1}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

and $$\cos 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

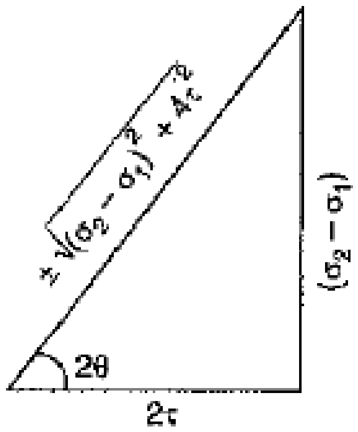


Fig. 3.12

Substituting the values of sin 2θ and cos 2θ in equation (3.13), the maximum and minimum shear stresses are obtained.

∴ Maximum shear stress is given by

$$(\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$= \pm \frac{\sigma_1 - \sigma_2}{2} \times \frac{(\sigma_2 - \sigma_1)}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \tau \times \frac{2\tau^2}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

$$= \pm \frac{(\sigma_1 - \sigma_2)^2}{2\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}} \pm \frac{2\tau^2}{\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}}$$

$$= \pm \frac{(\sigma_2 - \sigma_1)^2 + 4\tau^2}{2\sqrt{(\sigma_2 - \sigma_1)^3 + 4\tau^2}} = \pm \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}$$

$$\therefore \quad (\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_2 - \sigma_1)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \qquad \ldots(3.18)$$

The planes on which maximum shear stress is acting, are obtained after finding the two values of θ from equation (3.17). These two values of θ will differ by 90°.

The second method of finding the planes of maximum shear stress is to find first principal planes and principal stresses. Let θ_1 is the angle of principal plane with plane *BC* of Fig. 3.11 (*a*). Then the planes of maximum shear will be at $\theta_1 + 45°$ and $\theta_1 + 135°$ with the plane *BC* as shown in Fig. 3.12 (*a*).

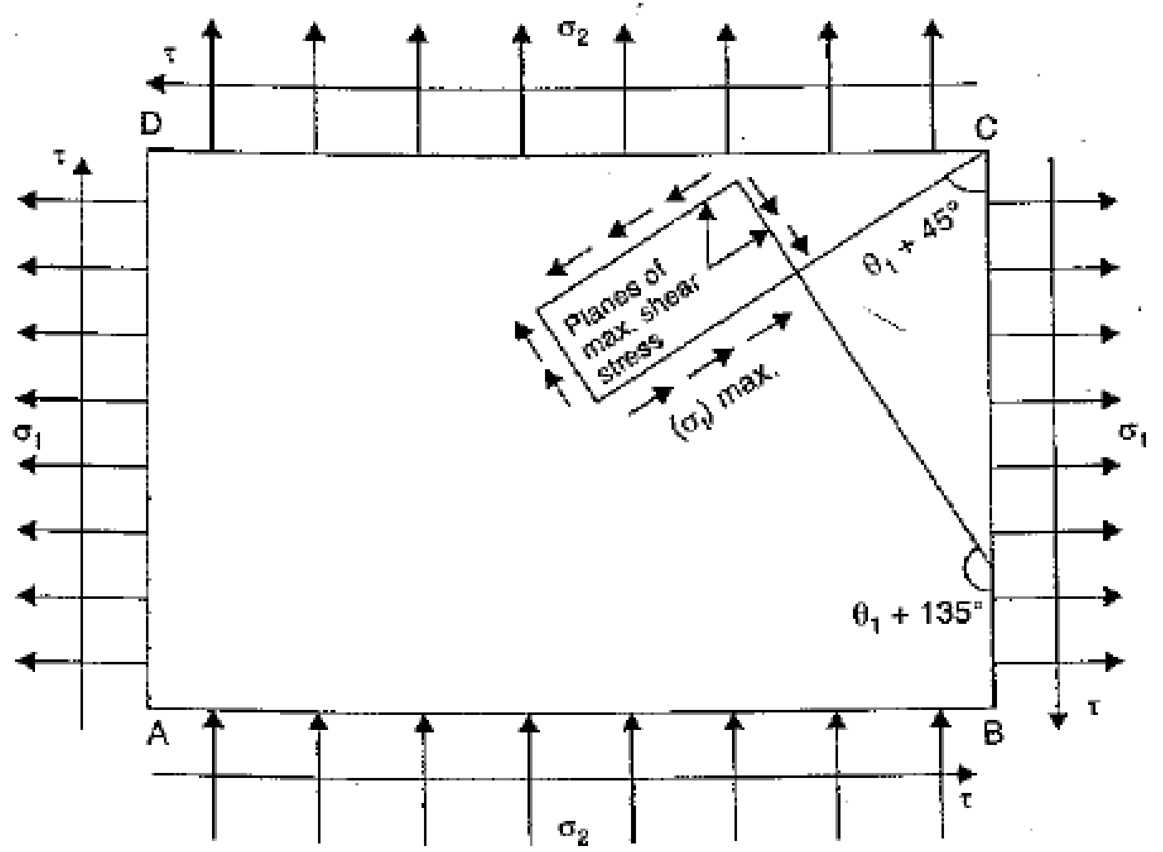


Fig. 3.12 (*a*)

Note. The above relations hold good when one or both the stresses are compressive.

Problem 3.11. *At a point within a body subjected to two mutually perpendicular directions, the stresses are 80 N/mm² tensile and 40 N/mm² tensile. Each of the above stresses is accompanied by a shear stress of 60 N/mm². Determine the normal stress, shear stress and resultant stress on an oblique plane inclined at an angle of 45° with the axis of minor tensile stress.*

Sol. Given :

Major tensile stress, $\sigma_1 = 80 \text{ N/mm}^2$

Minor tensile stress, $\sigma_2 = 40 \text{ N/mm}^2$

Shear stress, $\tau = 60 \text{ N/mm}^2$

Angle of oblique plane, with the axis of minor tensile stress,

$\theta = 45°$.

(*i*) *Normal stress* (σ_n)

Using equation (3.12),

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + \tau \sin 2\theta$$

$$= \frac{80 + 40}{2} + \frac{80 - 40}{2} \cos (2 \times 45°) + 60 \sin (2 \times 45°)$$

$$= 60 + 20 \cos 90° + 60 \sin 90°$$

$$= 60 + 20 \times 0 + 60 \times 1 \qquad (\because \cos 90° = 0)$$

$= 60 + 0 + 60 =$ **120 N/mm². Ans.**

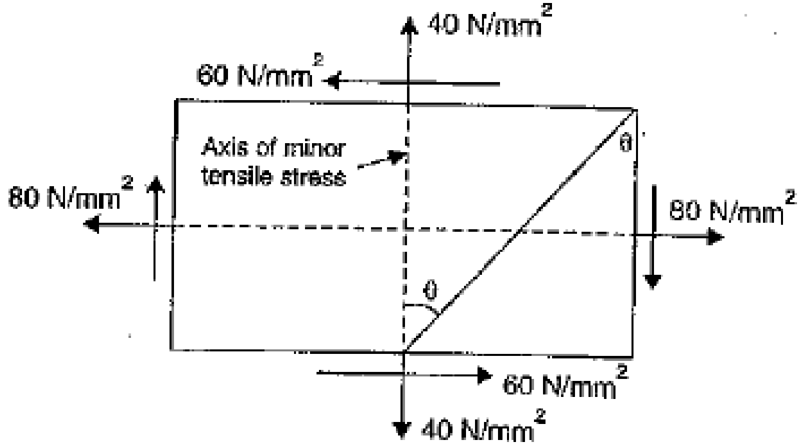


Fig. 3.13

(ii) Shear (or tangential) stress (σ_t)

Using equation (3.13),

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$= \frac{80 - 40}{2} \sin (2 \times 45°) - 60 \times \cos (2 \times 45°)$$

$$= 20 \times \sin 90° - 60 \cos 90°$$

$$= 20 \times 1 - 60 \times 0$$

$$= \mathbf{20\ N/mm^2}. \quad \textbf{Ans.}$$

(iii) Resultant stress (σ_R)

Using equation, $\sigma_R = \sqrt{{\sigma_n}^2 + {\sigma_t}^2}$

Problem 3.12. *A rectangular block of material is subjected to a tensile stress of 110 N/mm² on one plane and a tensile stress of 47 N/mm² on the plane at right angles to the former. Each of the above stresses is accompanied by a shear stress of 63 N/mm² and that associated with the former tensile stress tends to rotate the block anticlockwise. Find :*

(i) the direction and magnitude of each of the principal stress and

(ii) magnitude of the greatest shear stress. (AMIE, Summer 1983)

Sol. Given :

Major tensile stress, $\sigma_1 = 110 \text{ N/mm}^2$

Minor tensile stress, $\sigma_2 = 47 \text{ N/mm}^2$

Shear stress, $\tau = 63 \text{ N/mm}^2$

(*i*) Major principal stress is given by equation (3.15).

$$\therefore \text{ Major principal stress} = \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

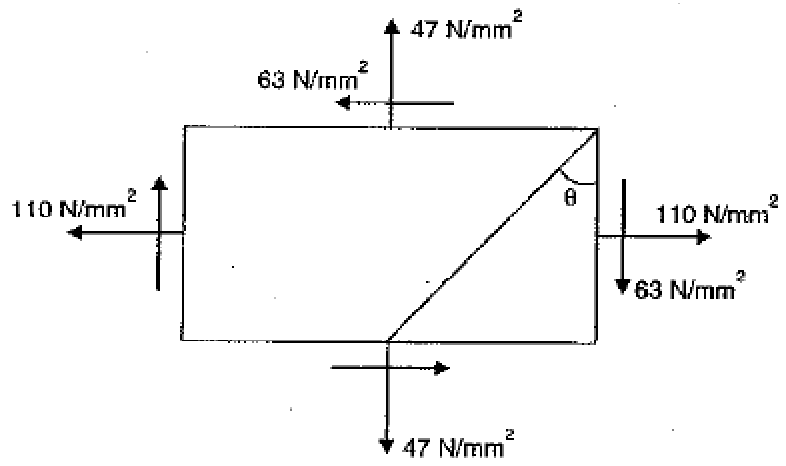


Fig. 3.14

$$= \frac{110 + 47}{2} + \sqrt{\left(\frac{110 - 47}{2}\right)^2 + 63^2}$$

$$= \frac{157}{2} + \sqrt{\left(\frac{63}{2}\right)^2 + (63)^2}$$

$$= 78.5 + \sqrt{31.5^2 + 63^2} = 78.5 + \sqrt{992.25 + 3969}$$

$$= 78.5 + 70.436 = \mathbf{148.936\ N/mm^2.} \quad \textbf{Ans.}$$

Minor principal stress is given by equation (3.16).

$\therefore$ Minor principal stress, $= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$

$$= \frac{110 + 47}{2} - \sqrt{\left(\frac{110 - 47}{2}\right)^2 + 63^2} = 78.5 - 70.436$$

$$= \mathbf{8.064\ N/mm^2.} \quad \textbf{Ans.}$$

The directions of principal stresses are given by equation (3.14).

∴ Using equation (3.14),

$$\tan 2\theta = \frac{2\tau}{\sigma_1 - \sigma_2} = \frac{2 \times 63}{110 - 47}$$

$$= \frac{2 \times 63}{63} = 2.0$$

$$\therefore \quad 2\theta = \tan^{-1} 2.0 = 63° \; 26' \text{ or } 243° \; 26'$$

$$\therefore \quad \theta = \mathbf{31° \; 43' \text{ or } 121° \; 43'.} \quad \textbf{Ans.}$$

(ii) *Magnitude of the greatest shear stress*

Greatest shear stress is given by equation (3.18).

Using equation (3.18),

$$(\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

$$= \frac{1}{2}\sqrt{(100 - 47)^2 + 4 \times 63^2}$$

$$= \frac{1}{2}\sqrt{63^2 + 4 \times 63^2} = \frac{1}{2} \times 63 \times \sqrt{5}$$

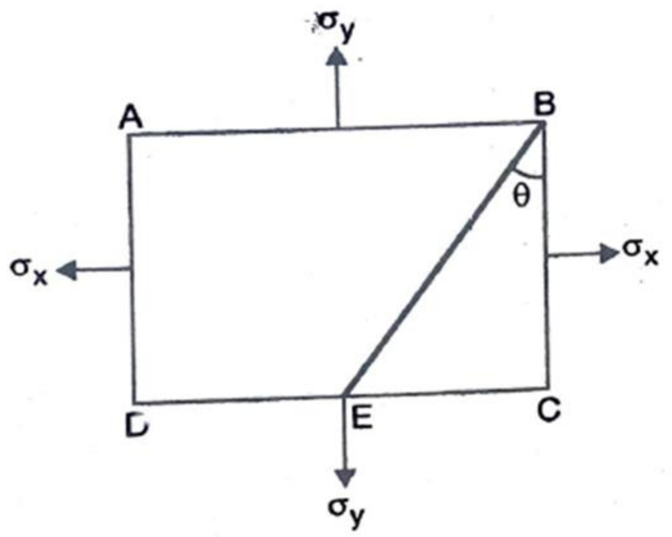
$$= \mathbf{70.436\ N/mm^2.} \quad \textbf{Ans.}$$

Determination of Stresses on Oblique Planes by Mohr's Circle Method i.e. Graphical Method :

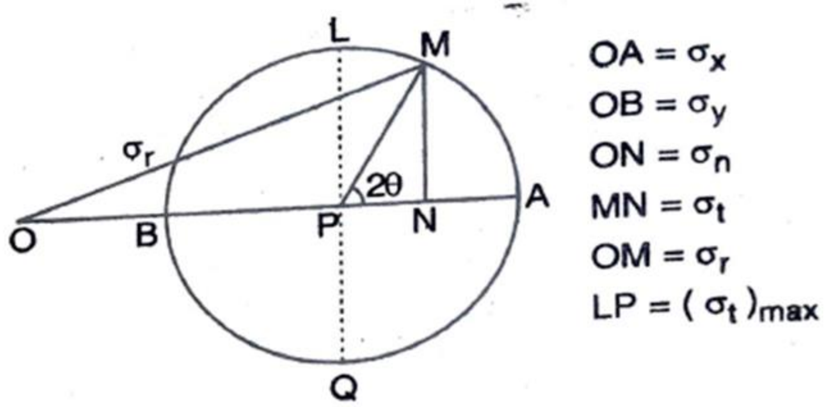
- Stresses on oblique planes can also be determined by graphical method called as Mohr's cir method.
- Mohr's circle method for various cases are explained as follows :

Pure Direct Stresses on Two Mutually Perpendicular Planes :

Case (a) : When both stresses are tensile :



(a) System of stresses



(b) Mohr's circle

1. If 'Q' is origin (to represent zero direct stress), stresses to the right of 'O' will repres tensile or positive stress and the stresses to the left of 'O' will represent compressive negative stresses.
2. Clockwise shear stresses on a direct stress plane will be represented by a vertical line abo the horizontal line and anticlockwise shear stress will be represented by a vertical line bel the horizontal line.

Method of drawing the Mohr's circle :

System of stresses is given in Fig. 7.8.1.1(a) and its corresponding Mohr's circle in Fig. 7.8.1.1(a) Method followed is :

1. Choose a point 'O' to represent zero direct stress and choose a suitable scale to represent stresses on the diagram.
2. Represent OA = σ_x and OB = σ_y on choosen scale.
3. Find the centre P of AB.
4. P as centre PA or PB as radius, draw a circle.

5. If θ is the angle of oblique plane from the plane of stress σ_x, draw a line at 2θ from PA a represented by line PM.

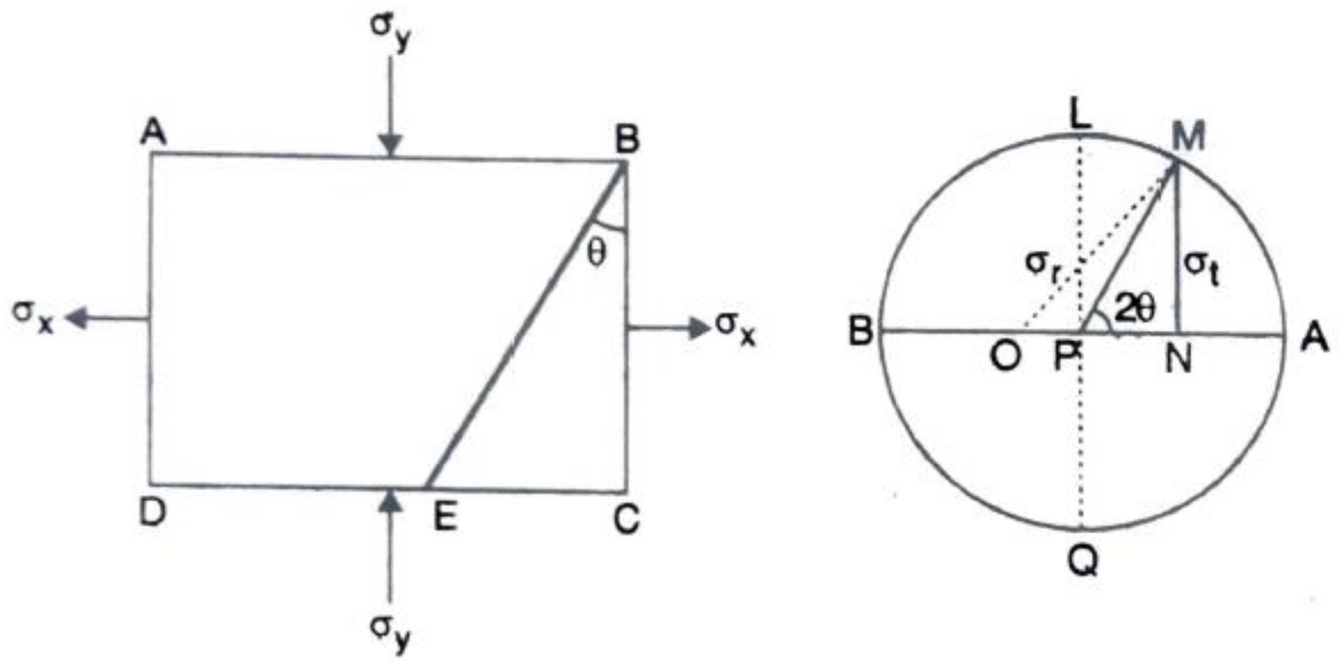
6. Draw perpendicular from M on OA as MN.

In the Mohr's diagram

ON represents σ_n and MN represents σ_t on oblique plane of stressed material.

OM represents the resultant stress σ_r on oblique plane.

Case (b) : Stresses σ_x is tensile and σ_y is compressive :



OA = σ_x (tensile)

OB = σ_y (compressive)

ON = σ_x

MN = σ_t

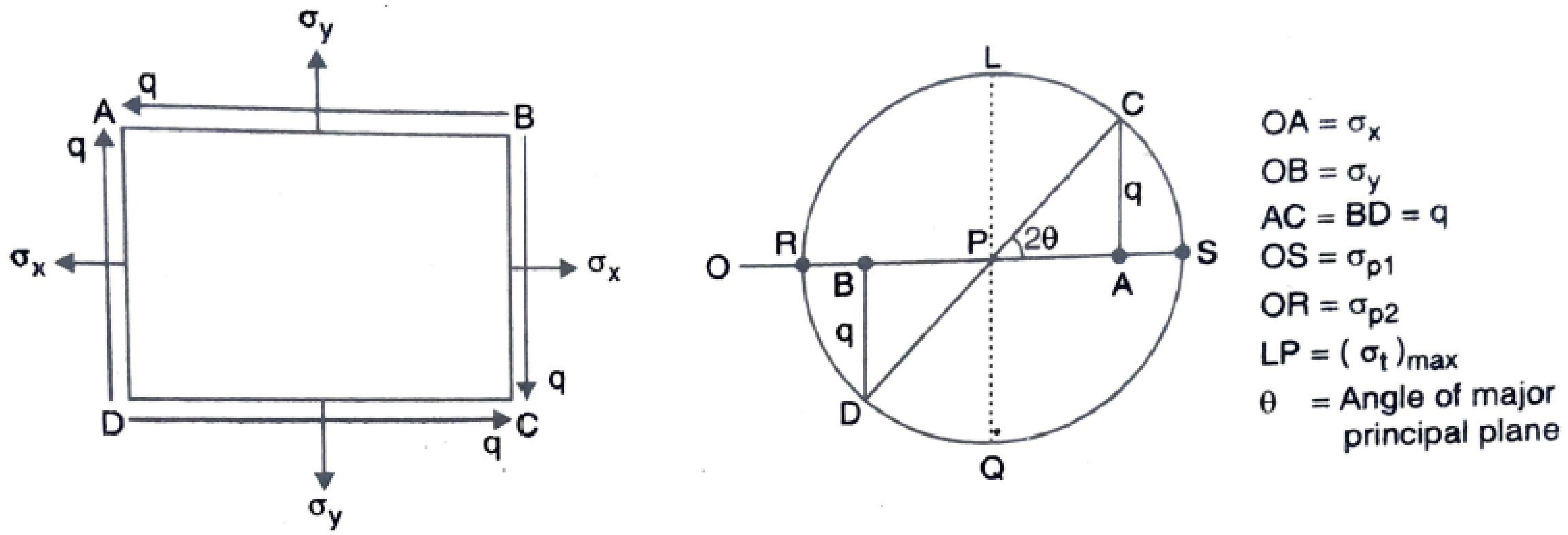
OM = σ_r

LP = $(\sigma_t)_{max}$

Method :

1. Choose point 'O' to represent zero direct stress.
2. Draw OA = σ_x (tensile) and OB = σ_y (compressive) to the left 'O' as per sign convention on suitable scale.
3. P is centre of AB. PA or PB as radius, describe a circle with P as centre.
4. Draw a line PM at an angle 2θ where θ is the angle of oblique plane from stress σ_x. Then σ_n = ON, q_t = MN and σ_r = OM

Determination of Principal Stresses for Two Dimensional Direct Stress System with Shear Stress by Mohr's Circle Method :



(a) System of stresses (b) Mohr's diagram

: Mohr's diagram for two dimensional stress system with shear stresses

- Fig. 7.9.1(a) shows the system of stresses and the Mohr's diagram for such a system is shown in Fig. 7.9.1(b). Method of drawing the diagram is as follow :
  1. Choose a point 'O' to represent zero direct stress.

2. Draw OA = σ_x and OB = σ_y, both tensile on suitable scale.

3. Draw a perpendicular at A upwards since shear stress on this plane is clockwise such that AC = q. Similarly, draw BD = q, downwards since shear stress on plane of σ_y is anticlockwise.

4. Join DC which cuts the horizontal line at P.

5. P as centre and PC = PD as radius, draw a circle which cuts the horizontal line at R and S.

Then the representation of principal and maximum shear stresses on this diagram is :

Major principal stress, $\sigma_{p_1} = OS$

Minor principal stress, $\sigma_{p_2} = OR$

Maximum shear stress, $(\sigma_t)_{max} = LP$

- Measure 2θ. Then θ and (θ + 90) will represent the planes of principal stresses.

Example 1 : At a point in a material two stresses on mutually perpendicular planes are $400 \, N/mm^2$ and $200 \, N/mm^2$, both compressive. Find the normal, tangential and resultant intensity of stress on an oblique plane at $\theta = 30^\circ$ from the plane of $400 \, N/mm^2$ by Mohr's diagram.

Solution :

Given : $\sigma_x = -400\ N/mm^2$, $\sigma_y = -200\ N/mm^2$, $\theta = 30^\circ$

Refer Fig. 7.10.1(a) and (b).

System of stresses :

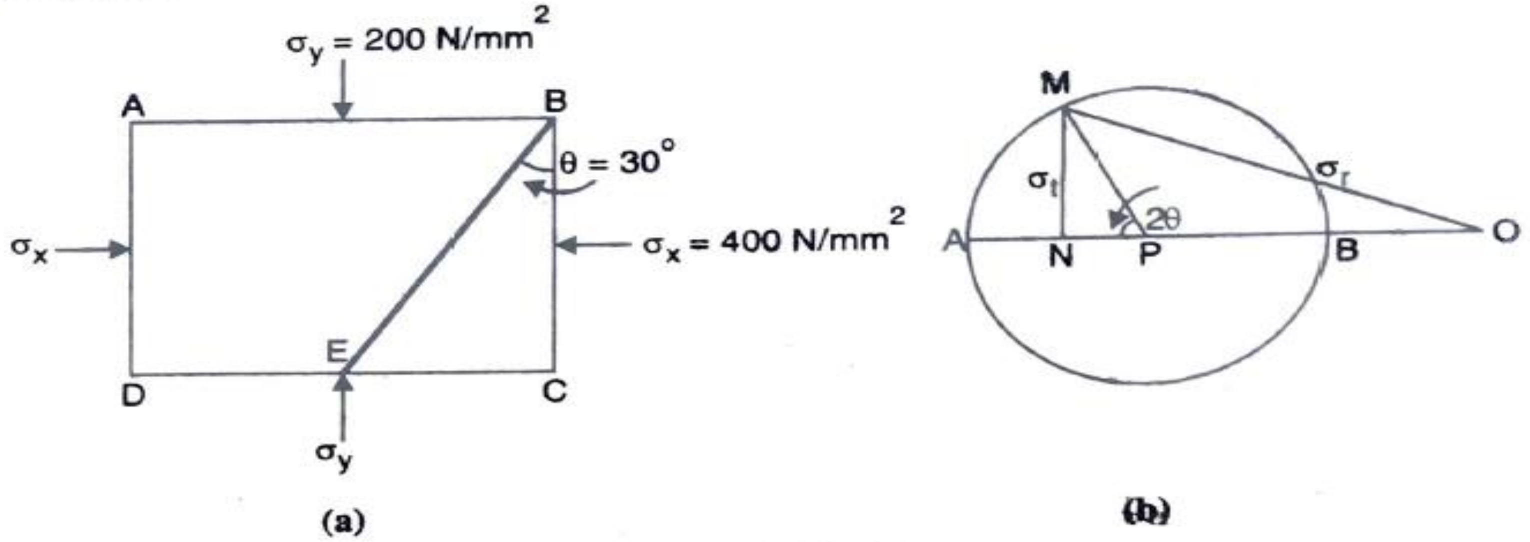


Fig. 7.10.1 : Mohr's diagram

Method :

1. Draw OA = 400 N/mm^2 = σ_x on a choosen scale of 1 cm = 50 N/mm^2 to the left of 'O'

2. Draw OB = σ_y = 200 N/mm^2 (compressive).

3. Bisect BA at P.

4. P as centre and PA as radius, draw a circle.

5. From P, draw a line PM at $2\theta = 60^\circ$ where $\theta = 30^\circ$ of the oblique plane from σ_x

6. Draw PN perpendicular to OA.

By measurement,

Normal stress, σ_n = ON = **360 N/mm^2** (compressive)

Tangential stress, σ_t = MN = **87 N/mm^2**

Resultant stress, σ_r = OM = **370 N/mm^2**

Example 2 : Find the principal stresses and principal planes for a rectangular block subjected to stresses as shown in Fig. 7 10 2 by Mohr's diagram.

Also find the magnitude of maximum shear stress.

Also check the results by analytical method.

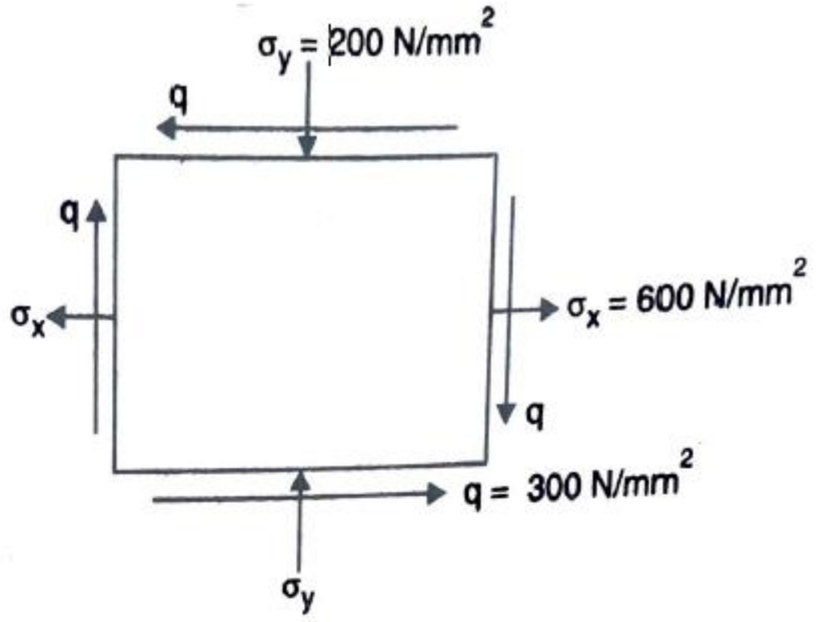


Fig. 7.10.2

Solution :

Given : $\sigma_x = 600\ N/mm^2$ (Tensile)

$\sigma_y = -200\ N/mm^2$ (Compressive)

q = shear stress = 300 N/mm²

The Mohr's diagram can be drawn as shown in Fig. 7.10.3 on a scale as 1 cm = 100 N/mm^2

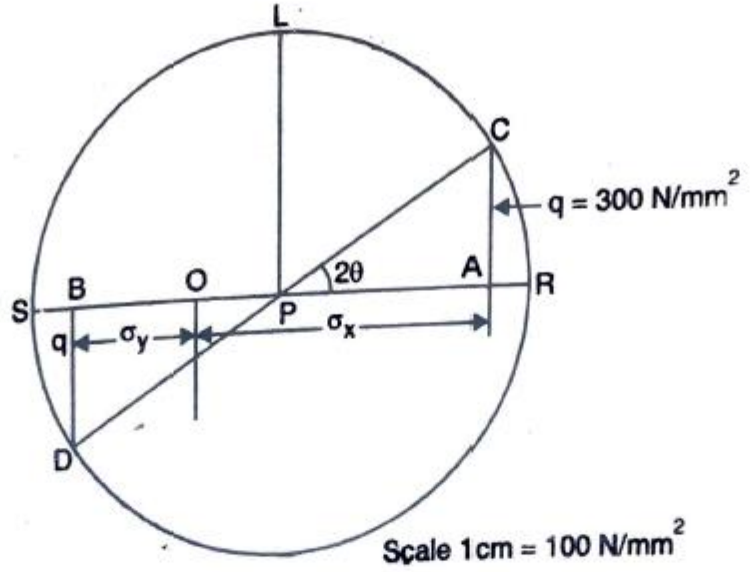


Fig. 7.10.3 : Mohr's diagram

Method of drawing Mohr's diagram :

1. Choose point 'O' to represent zero direct stress.
2. Draw σ_x = OA = 600 N/mm^2 (tensile) and σ_y = OB = 200 N/mm^2 (compressive) on suitable scale.
3. Draw q = AC = 300 N/mm^2 on σ_x and q = BD = 300 N/mm^2 on σ_y.

4. Join CD which intersects OA at P.

5. P as centre and PC as radius, draw a circle which cuts the horizontal line at R and S.

6. From P draw perpendicular LP,

Then by measurement, we get,

Major principal stress, σ_{p_1} = OR = 700 N/mm^2 (tensile)

Minor principal stress, σ_{p_2} = OS = 300 N/mm^2 (compressive)

$2\theta = 37^\circ$; $\therefore$ $\theta = 18.5^\circ$

$\therefore$ Principal planes are at θ = 18.5° or 108.5°

Maximum shear stress $(\sigma_t)_{max}$ = LP = 490 N/mm^2

Check by Analytical method :

Principal stresses are given as :

Major principal stress, $\sigma_{p_1} = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + q^2}$

$$= \left(\frac{600-200}{2}\right) + \sqrt{\left[\frac{600-(-200)}{2}\right]^2 + 300^2}$$

$$= 200 + 500 = \mathbf{700\ N/mm^2\ (tensile)}$$

Minor principle stress, $\sigma_{p_2} = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + q^2}$

$$= \left(\frac{600-200}{2}\right) - \sqrt{\left[\frac{600-(-200)}{2}\right]^2 + 300^2}$$

$$= 200 - 500 = \mathbf{-300\ N/mm^2\ (compressive)}$$

Principal plane :

$$\tan 2\theta = \frac{2q}{\sigma_x - \sigma_y} = \frac{2 \times 300}{600 - (-200)} = 0.75$$

$$2\theta = 36.86^\circ \text{ or } 216.86^\circ$$

$$\therefore \quad \theta = 18.43^\circ \text{ or } 108.43^\circ$$

...**Ans.**

Maximum shear stress,

$$(\sigma_t)_{max} = \frac{\sigma_{p_1} - \sigma_{p_2}}{2} = \frac{700 - (-300)}{2}$$

$$= \mathbf{500\ N/mm^2}$$

...**Ans.**

Note : $q_{max} = 490\ N/mm^2$ by graphical method is due to error in measurement

Example 3 : At a point in a strained material there are two mutually perpendicular stresses of 30 MPa and 70 MPa both tensile. They are accompanied by a shear stress of 20 MPa. Determine principal plane and principal stresses. Use Mohr's stress circle method only. **(S-08, 4 Marks)**

Solution :

Given : σ_x = 30 MPa (tensile), σ_y = 70 MPa (tensile), q = shear stress = 20 MPa

Draw the Fig. 7.10.4(a) from the given data for better undrstanding.

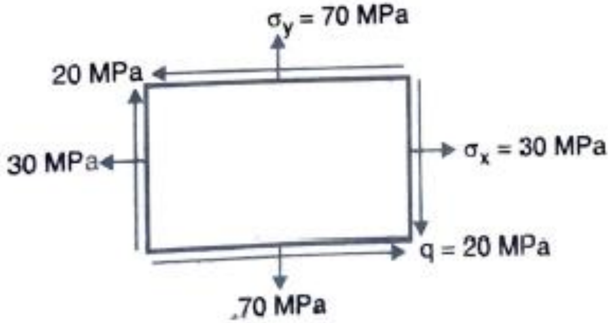


Fig. 7.10.4(a)

Method of drawing Mohr's diagram :

1. Choose point 'O' to represent zero direct stress.
2. Take the scale as 1 cm = 10 MPa and then draw σ_x = OA = 3 cm for 30 MPa (tensile) and σ_y = OB = 7 cm for 70 MPa (tensile) on same side of 'O' as shown in Fig. 12.
3. Draw a perpendicular at A upwards since shear stress on this plane is clockwise such that

AD = q. Similarly, draw BC = q downwards since shear stress on plane of σ_y is anticlockwise.

4. Join DC which cuts the horizontal line at P.

5. P as a centre and PC = PD as a radius, draw a circle which cuts the horizontal line at R and S.

6. See the Fig. 7.10.4(b).

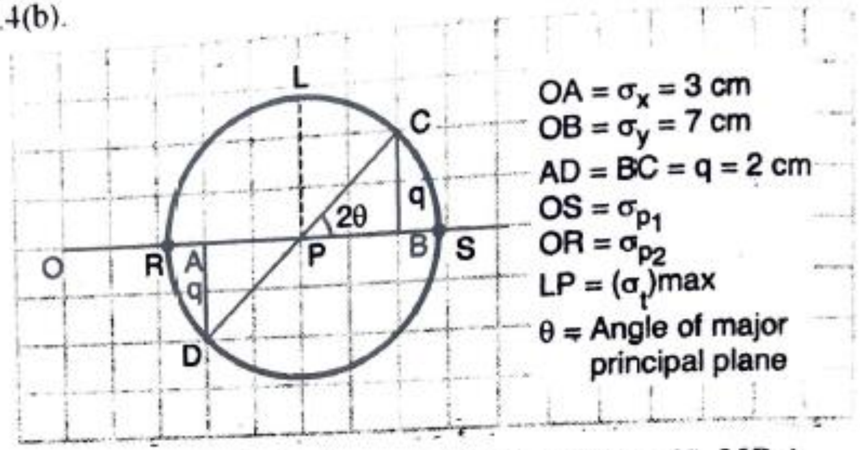


Fig. 7.10.4(b) : Mohr's diagram (scale 1 cm = 10 MPa)

To find principal stresses and planes :

- Measure 'OS' which gives the major principal stress (σ_{p_1})

$\therefore \quad \sigma_{p_1} = l(\text{OS}) \times \text{scale} = 7.8 \times 10 = \textbf{78 MPa}$ ...Ans.

Problem 3.13. *Direct stresses of 120 N/mm² tensile and 90 N/mm² compression exist on two perpendicular planes at a certain point in a body. They are also accompanied by shear stress on the planes. The greatest principal stress at the point due to these is 150 N/mm².*

(a) *What must be the magnitude of the shearing stresses on the two planes ?*

(b) *What will be the maximum shearing stress at the point ?*

Sol. Given :

Major tensile stress, $\sigma_1 = 120 \text{ N/mm}^2$

Minor compressive stress, $\sigma_2 = -90 \text{ N/mm}^2$ (Minus sign due to compression)

Greatest principal stress $= 150 \text{ N/mm}^2$

(a) Let τ = Shear stress on the two planes.

Using equation (3.15) for greatest principal stress, we get

$$\text{Greatest principal stress} = \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

or

$$150 = \frac{120 + (-90)}{2} + \sqrt{\left(\frac{120 - (-90)}{2}\right)^2 + \tau^2}$$

$$= \frac{120 - 90}{2} + \sqrt{\left(\frac{120 + 90}{2}\right)^2 + \tau^2}$$

$$= 15 + \sqrt{105^2 + \tau^2}$$

or $$150 - 15 = \sqrt{105^2 + \tau^2}$$

or $$135 = \sqrt{105^2 + \tau^2}$$

Squaring both sides, we get

$$135^2 = 105^2 + \tau^2$$

or $$\tau^2 = 135^2 - 105^2 = 18225 - 11025 = 7200$$

$\therefore$ $$\tau = \sqrt{7200} = 84.853 \text{ N/mm}^2. \quad \textbf{Ans.}$$

(b) Maximum shear stress at the point

Using equation (3.18) for maximum shear stress,

$$(\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}$$

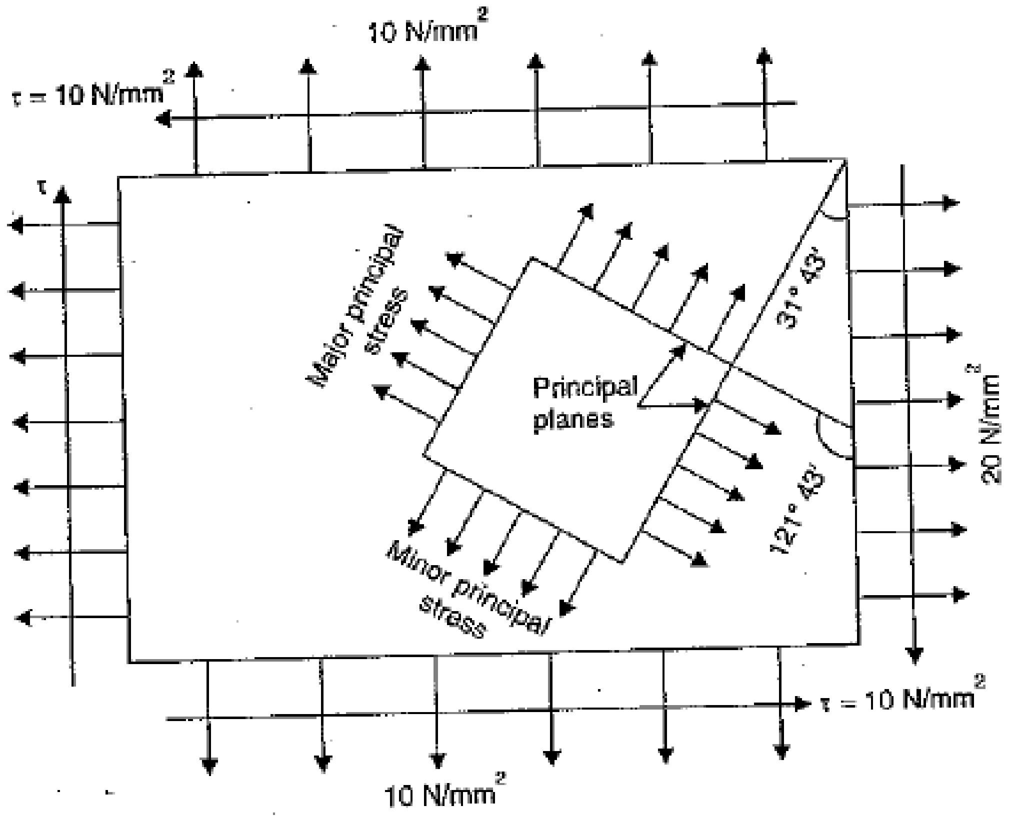
$$= \frac{1}{2}\sqrt{[120 - (-90)]^2 + 4 \times 7200} \qquad (\because \tau^2 = 7200)$$

$$= \frac{1}{2}\sqrt{210^2 + 28800} = \frac{1}{2}\sqrt{44100 + 28800} = \frac{1}{2} \times 270$$

$$= 135 \text{ N/mm}^2. \quad \textbf{Ans.}$$

Problem 3.14. *At a certain point in a strained material, the stresses on two planes, at right angles to each other are 20 N/mm² and 10 N/mm² both tensile. They are accompanied by a shear stress of a magnitude of 10 N/mm². Find graphically or otherwise, the location of principal planes and evaluate the principal stresses.* (AMIE, Summer 1984)

Sol. Given :



Major tensile stress, $\sigma_1 = 20$ N/mm²

Minor tensile stress, $\sigma_2 = 10$ N/mm²

Shear stress, $\tau = 10$ N/mm²

Location of principal planes

The location of principal planes is given by equation (3.14).

Using equation (3.14),

$$\tan 2\theta = \frac{2\tau}{\sigma_1 - \sigma_2} = \frac{2 \times 10}{20 - 10} = \frac{2 \times 10}{10} = 2.0$$

$$\therefore \quad 2\theta = \tan^{-1} 2.0 = 63° \, 26' \text{ or } 243° \, 26'$$

or $$\theta = 31° \, 43' \text{ or } 121° \, 43'. \quad \textbf{Ans.}$$

Magnitude of principal stresses

The major principal stress is given by equation (3.15)

∴ Major principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2} = \frac{20 + 10}{2} + \sqrt{\left(\frac{20 - 10}{2}\right)^2 + 10^2}$$

$$= 15 + \sqrt{5^2 + 100} = 15 + \sqrt{25 + 100} = 15 + \sqrt{125} = 15 + 11.18$$

$$= \textbf{26.18 N/mm}^2. \quad \textbf{Ans.}$$

The minor principal stress is given by equation (3.16).

∴ Minor principal stress

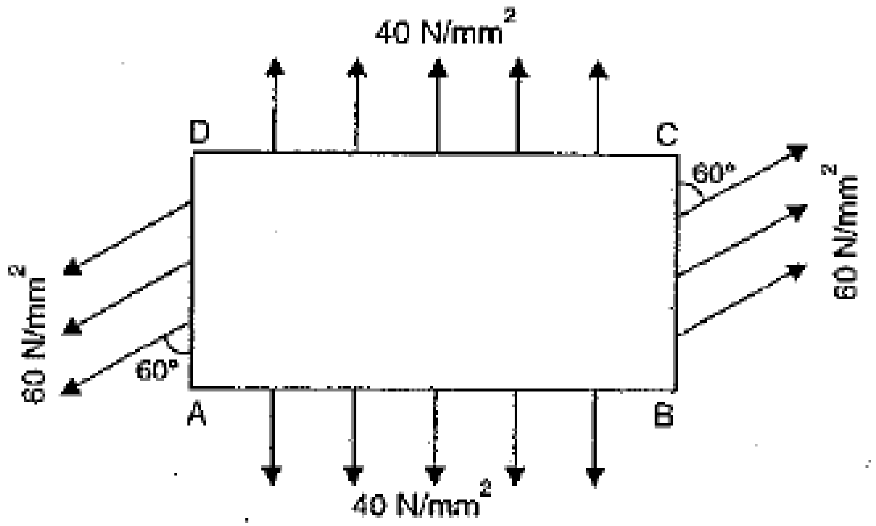
$$= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{20 + 10}{2} - \sqrt{\left(\frac{20 - 10}{2}\right)^2 + 10^2}$$

$$= 15 - 11.18 = \textbf{3.82 N/mm}^2. \quad \textbf{Ans.}$$

Problem 3.15. *A point in a strained material is subjected to the stresses as shown in Fig. 3.15.*

Locate the principal planes, and evaluate the principal stresses.



Sol. Given :

The stress on the face BC or AD is not normal. It is inclined at an angle of 60° with face BC or AD. This stress can be resolved into two components *i.e.*, normal to the face BC (or AD) and along the face BC (or AD).

∴ Stress normal to the face BC or AD

$$= 60 \times \sin 60° = 60 \times 0.866 = 51.96 \text{ N/mm}^2$$

Stress along the face BC or AD

$$= 60 \times \cos 60° = 60 \times 0.5 = 30 \text{ N/mm}^2$$

The stress along the face BC or AD is known as shear stress. Hence $\tau = 30 \text{ N/mm}^2$. Due to complementary shear stress the face AB and CD will also be subjected to shear stress of 30 N/mm^2. Now the stresses acting on the material are shown in Fig. 3.16.

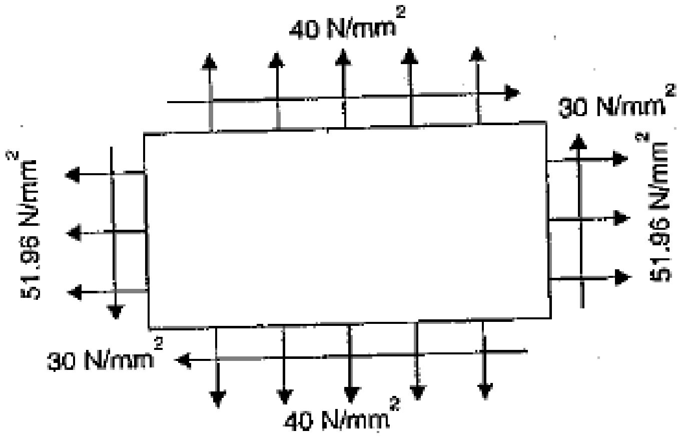


Fig. 3.16

Major tensile stress, $\sigma_1 = 51.96 \text{ N/mm}^2$

Shear stress, $\tau = 30 \text{ N/mm}^2$

Location of principal planes

Let θ = Angle, which one of the principal planes make with the stress of 40 N/mm².

The location of the principal planes is given by the equation (3.14).

Using equation (3.14), we get

$$\tan 2\theta = \frac{2\tau}{\sigma_1 - \sigma_2} = \frac{2 \times 30}{51.96 - 40} = 4.999$$

$$\therefore \quad 2\theta = \tan^{-1} 4.999 = 78° \, 42' \text{ or } 258° \, 42'$$

or $\quad \mathbf{\theta = 39° \, 21' \text{ or } 129° \, 21'.}$ **Ans.**

Principal stress

The major principal stress is given by equation (3.15).

$\therefore$ Major principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{51.96 + 40}{2} + \sqrt{\left(\frac{51.96 - 40}{2}\right)^2 + 30^2}$$

$$= 45.98 + 30.6$$

$$= \mathbf{76.58\ N/mm^2.}\quad \textbf{Ans.}$$

The minor principal stress is given by equation (3.16).

∴ Minor principal stress

$$= \frac{\sigma_1 + \sigma_2}{2} - \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + \tau^2}$$

$$= \frac{51.96 + 40}{2} - \sqrt{\left(\frac{51.96 - 40}{2}\right)^2 + 30^2}$$

$$= 45.98 - 30.6$$

$$= \mathbf{15.38\ N/mm^2.}\quad \textbf{Ans.}$$

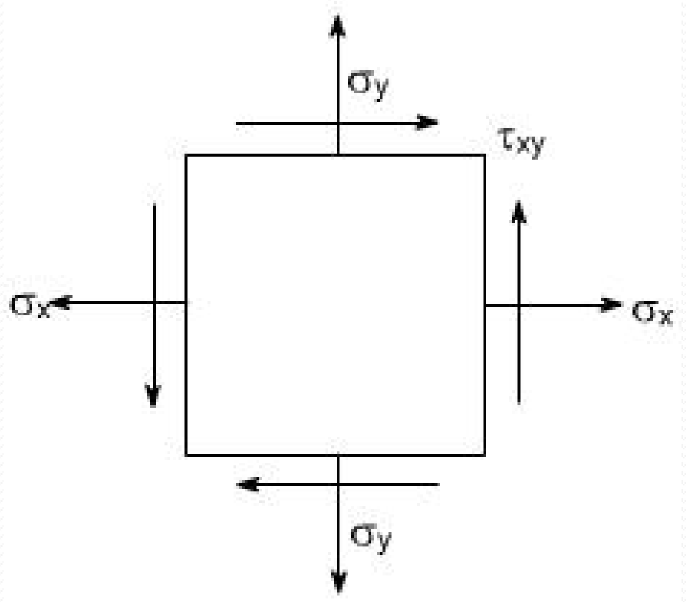
Different Theories of Failure :

These are five different theories of failures which are generally used

(a) Maximum Principal stress theory (due to Rankine)
(b) Maximum shear stress theory (Guest - Tresca)
(c) Maximum Principal strain (Saint - venant) Theory
(d) Total strain energy per unit volume (Haigh) Theory
(e) Shear strain energy per unit volume Theory (Von – Mises & Hencky

(a) Maximum Principal stress theory :

- This theory assume that when the maximum principal stress in a complex stress system reaches the elastic limit stress in a simple tension, failure will occur. Therefore the criterion for failure would be σ1 = σ yp For a two dimensional complex stress system σ 1 is expressed as Where σ x, σ y and σ xy are the stresses in the any given complex stress system.

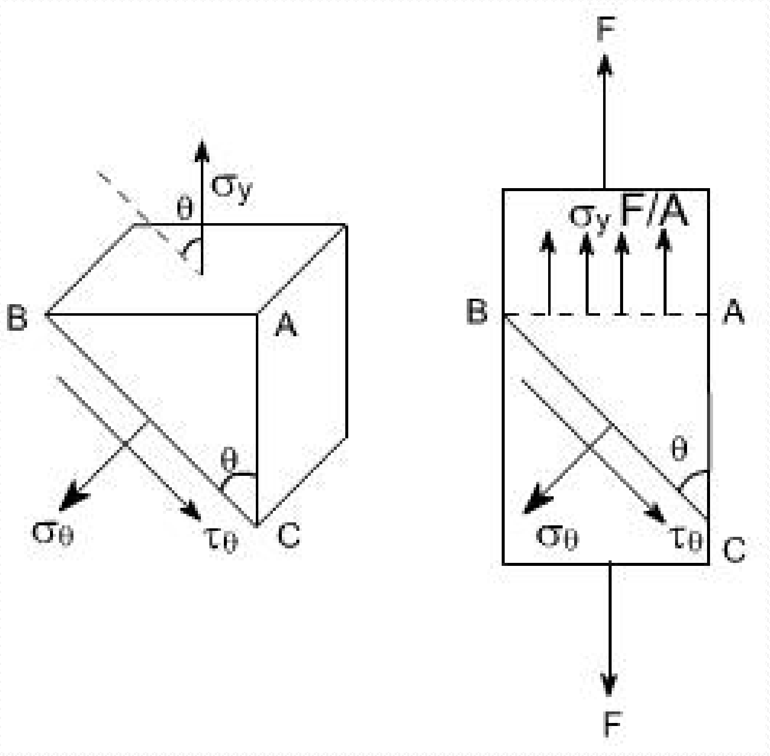


$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4.\tau_{xy}^2}$$

$$= \sigma_{yp}$$

(b) Maximum shear stress theory:

- This theory states that the failure can be assumed to occur when the maximum shear stress in the complex stress system is equal to the value of maximum shear stress in simple tension.
- The criterion for the failure may be established as given below :



(b) Maximum shear stress theory:

$$\sigma_\theta = \sigma_y \sin^2\theta$$

$$\tau_\theta = \frac{1}{2}\sigma_y \sin 2\theta$$

$$\tau_\theta\Big|_{maxm} = \frac{1}{2}\sigma_y \quad \text{or}$$

$$\tau_{maxm} = \frac{1}{2}\sigma_{yp}$$

whereas for the two dimentional complex stress system

$$\tau_{maxm} = \left(\frac{\sigma_1 - \sigma_2}{2}\right)$$

where σ_1 = maximum principle stress

σ_2 = minimum principal stress

$$\text{so} \quad \frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy}$$

$$\frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\sigma_{yp} \Rightarrow \sigma_1 - \sigma_2 = \sigma_y$$

$$\Rightarrow \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2 xy} = \sigma_{yp}$$

becomes the criterion for the failure.

(c) Maximum Principal strain theory :

- This Theory assumes that failure occurs when the maximum strain for a complex state of stress system becomes equals to the strain at yield point in the tensile test for the three dimensional complex state of stress system.
- For a 3 - dimensional state of stress system the total strain energy U_t per unit volume in equal to the total work done by the system and given by the equation

(c) Maximum Principal strain theory :

$$U_t = 1/2\sigma_1 \epsilon_1 + 1/2\sigma_2 \epsilon_2 + 1/2\sigma_3 \epsilon_3$$

substituting the values of ϵ_1, ϵ_2 and ϵ_3

$$\epsilon_1 = \frac{1}{E}[\sigma_1 - \gamma(\sigma_2 + \sigma_3)]$$

$$\epsilon_2 = \frac{1}{E}[\sigma_2 - \gamma(\sigma_1 + \sigma_3)]$$

$$\epsilon_3 = \frac{1}{E}[\sigma_3 - \gamma(\sigma_1 + \sigma_2)]$$

Thus, the failure criterion becomes

$$\left(\frac{\sigma_1}{E} - \gamma\frac{\sigma_2}{E} - \gamma\frac{\sigma_3}{E}\right) = \frac{\sigma_{yp}}{E}$$

or

$$\boxed{\sigma_1 - \gamma\sigma_2 - \gamma\sigma_3 = \sigma_{yp}}$$

(d) Total strain energy per unit volume theory

- The theory assumes that the failure occurs when the total strain energy for a complex state of stress system is equal to that at the yield point a tensile test.
- Therefore, the failure criterion becomes
- It may be noted that this theory gives fair by good results for ductile materials.

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\gamma(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] = \frac{\sigma_{yp}^2}{2E}$$

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\gamma(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_{yp}^2$$

(e) Maximum shear strain energy per unit volume theory :

This theory states that the failure occurs when the maximum shear strain energy component for the complex state of stress system is equal to that at the yield point in the tensile test.

$$\frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = \frac{\sigma_{yp}^2}{6G}$$

Where G = shear modulus of regidity

$$\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] = 2\sigma_{yp}^2$$

(e) Maximum shear strain energy per unit volume theory :

This theory states that the failure occurs when the maximum shear strain energy component for the complex state of stress system is equal to that at the yield point in the tensile test.

$$\frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=\frac{\sigma_{yp}^2}{6G}$$

Where G = shear modulus of regidity

$$\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]=2\sigma_{yp}^2$$

(ii) Distortional or Deviatoric state of stress

This is the distortion strain energy for a complex state of stress, this is to be equaled to the maximum distortion energy in the simple tension test. In order to get we may assume that one of the principal stress say (σ_1) reaches the yield point (σ_{yp}) of the material. Thus, putting in above equation $\sigma_2 = \sigma_3 = 0$ we get distortion energy for the simple test i.e

$$U_{distortion} = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$U_d = \frac{2\sigma_1^2}{12G}$$

Futher $\sigma_1 = \sigma_{yp}$

Thus, $\boxed{U_d = \frac{\sigma_{yp}^2}{6G}}$ for a simple tension test.